

DEFORMATIONS OF LAGRANGIAN TYPE SUBMANIFOLDS INSIDE G_2 MANIFOLDS

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Dedicated to the memory of Ruth I. Michler

ABSTRACT. 3-dimensional Harvey Lawson submanifolds were introduced in an earlier paper by Akbulut-Salur, [1] as examples of Lagrangian-type manifolds inside G_2 manifold. In this paper, we first show that the space of deformations of a smooth, compact, orientable Harvey-Lawson submanifold HL in a G_2 manifold M can be identified with the direct sum of the space of smooth functions and closed 2-forms on HL . We then introduce a new class of Lagrangian-type 4-dimensional submanifolds inside G_2 , call them RS submanifolds and prove that the space of deformations of a smooth, compact, orientable RS submanifold in a G_2 manifold M can be identified with closed 3-forms on RS .

1. INTRODUCTION

It is well-known that a smooth symplectic manifold N^{2n} is equipped with a closed, nondegenerate differential 2-form ω . An n -dimensional submanifold L^n of N^{2n} is called Lagrangian if the restriction of ω to L is zero. Lagrangian submanifolds and their deformations have important applications in symplectic geometry and mathematical physics. In particular, they play a role in establishing the correspondence between “Calabi-Yau mirror pairs” in string theory via the Fukaya category.

Let (M, φ) be a G_2 manifold with calibration 3-form φ . As with the symplectic 2-form ω , the calibration 3-form φ on a G_2 manifold is closed and nondegenerate. Moreover, the Hodge dual, $\star\varphi$, of φ is also closed and nondegenerate. Therefore G_2 manifolds provide a natural setting in which one can search for Lagrangian-type submanifolds that correspond to φ and $\star\varphi$. We call them Harvey-Lawson (HL) and RS manifolds, respectively. For more on the definition and properties of HL manifolds we refer the reader to [1]. One can study these submanifolds and their deformation spaces and moreover, search for relations by using Fukaya categories. Through these results, we hope to use the relationship between Calabi-Yau 3-folds and G_2 manifolds to answer questions about the mirror symmetry of Calabi-Yau 3-folds and existence of calibrated submanifolds, [2], [3], [4], [6], [12], [13]. This will be our motivation for the future directions, [8], [11].

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In this paper we define these Lagrangian-type submanifolds of a G_2 manifold M . We then describe some properties of these objects and prove the following theorems.

Theorem 1.1. *The space of infinitesimal deformations of a smooth, compact, orientable 3-dimensional Harvey-Lawson submanifold HL in a G_2 manifold M within the class of HL submanifolds is infinite-dimensional. The deformation space can be identified with the direct sum of the spaces of smooth functions and closed differential 2-forms on HL .*

Theorem 1.2. *The space of all infinitesimal deformations of a smooth, compact, orientable 4-dimensional RS submanifold in a G_2 manifold M within the class of RS submanifolds is infinite-dimensional and can be identified with closed differential 3-forms on RS .*

Remark 1.3. Note that M does not have to be a manifold with G_2 holonomy for these theorems. Theorem 1.1 also holds when M is a manifold with a closed G_2 structure ϕ and Theorem 1.2 can be extended to the case when M is a manifold with a co-closed G_2 structure ϕ .

2. DEFORMATIONS OF HL SUBMANIFOLDS

In this section, we study infinitesimal deformations of HL submanifolds. First, let's recall some basic definitions. For more details about G_2 manifolds, we refer the reader to [5], [7], and [9].

A manifold with G_2 structure is a smooth seven-dimensional manifold M such that the structure group of M reduces to the exceptional Lie group G_2 . Equivalently, since G_2 can be defined as the automorphism group of \mathbb{R}^7 that preserves the 3-form

$$\varphi_0 = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356}$$

where $e^{123} = dx^1 \wedge dx^2 \wedge dx^3$ and (x_1, \dots, x_7) are coordinates on \mathbb{R}^7 , we can define a G_2 structure in the following manner.

Definition 2.1. A manifold with G_2 structure is a smooth 7-dimensional manifold M equipped with a nondegenerate 3-form $\varphi \in \Omega^3(M)$ such that at any point $p \in M$,

$$(T_p M, \varphi_p) \cong (\mathbb{R}^7, \varphi_0).$$

This definition implies that on a local chart of a manifold with G_2 structure (M, φ) , up to quadratic terms the 3-form φ coincides with the form $\varphi_0 \in \Omega^3(\mathbb{R}^7)$. The 3-form φ determines a metric g and a cross product \times on M given by

$$\varphi(u, v, w) = g(u, v \times w), \quad u, v, w \in TM.$$

In this paper, we write this metric as $g(\cdot, \cdot) = \langle \cdot, \cdot \rangle$ for simplicity.

Definition 2.2. Suppose (M, φ) is a manifold with G_2 structure. We call (M, φ) a G_2 manifold if φ is torsion-free with respect to the Levi-Civita connection. Equivalently, we could say that M is a G_2 manifold if M has holonomy contained in the Lie group G_2 .

The torsion-free condition is equivalent to the condition that the form φ is closed and co-closed, i.e.

$$d\varphi = d \star \varphi = 0.$$

We call φ and $\star\varphi$ the calibration 3-form and 4-form, respectively, as they define *calibrated submanifolds*; manifolds that are volume-minimizing in their homology class.

Definition 2.3. Let (M, φ) be a G_2 manifold with calibration 3-form φ . A 4-dimensional submanifold $C \subset M$ is called *coassociative* if $\varphi|_C = 0$. A 3-dimensional submanifold $A \subset M$ is called *associative* if $\varphi|_A \equiv d\text{vol}(A)$.

Note that the condition $\varphi|_A \equiv d\text{vol}(A)$ is equivalent to the condition that $\chi|_A \equiv 0$, where $\chi \in \Omega^3(M, TM)$ is the tangent bundle-valued 3-form defined by the identity

$$\langle \chi(u, v, w), z \rangle = \star\varphi(u, v, w, z).$$

Remark 2.4. The equivalence of $\varphi|_A = d\text{vol}$ and $\langle \chi|_A, \chi|_A \rangle = 0$ follows from the associator equality which was shown in Harvey-Lawson, [9]:

$$\varphi(u, v, w)^2 + \frac{1}{4}|\chi(u, v, w)|^2 = |u \wedge v \wedge w|^2.$$

Definition 2.5. A *Harvey-Lawson manifold* is a 3-dimensional submanifold $HL \subset M$ of a G_2 manifold such that

$$\varphi|_{HL} = 0.$$

Equivalently, this is defined by $\langle \chi|_{HL}, \chi|_{HL} \rangle = 1$.

Remark 2.6. Again, the associator equality, [9], gives the equivalence of the conditions $\varphi|_{HL} = 0$ and $\langle \chi|_{HL}, \chi|_{HL} \rangle = 1$.

Remark 2.7. One can obtain G_2 manifolds from Calabi-Yau manifolds in the following way. Let (N, ω, Ω) be a complex 3-dimensional Calabi-Yau manifold with Kähler form ω and a nonvanishing holomorphic $(3,0)$ -form Ω . Then the direct product $N^6 \times S^1$ has holonomy group $SU(3)$ which is a subset of G_2 . Therefore $N^6 \times S^1$ is a G_2 manifold. In this particular case, $\varphi = \text{Re } \Omega + \omega \wedge dt$. For the noncompact case $N^6 \times \mathbb{R}$ is also a G_2 manifold (with reduced holonomy). Using these structures, it is easy to show that all special Lagrangian submanifolds (with phase $\theta = \frac{\pi}{2}$) of the Calabi-Yau manifold N will be Harvey-Lawson submanifolds of $N^6 \times S^1$ or $N^6 \times \mathbb{R}$.

Similar to the tangent bundle-valued 3-form χ on a G_2 manifold, there is also a tangent bundle-valued 2-form, which is just the cross product of M .

Definition 2.8. Let (M, φ) be a G_2 manifold. Then $\psi \in \Omega^2(M, TM)$ is the tangent bundle-valued 2-form defined by the identity

$$\langle \psi(u, v), w \rangle = \varphi(u, v, w) = \langle u \times v, w \rangle.$$

We can also express the tangent bundle-valued forms χ and ψ in local coordinates as above for φ . More generally, if e_1, \dots, e_7 is any local orthonormal frame with dual frame e^1, \dots, e^7 , by definition we can write χ and ψ in coordinates as

$$\begin{aligned}
\chi = & (e^{256} + e^{247} + e^{346} - e^{357})e_1 \\
& + (-e^{156} - e^{147} - e^{345} - e^{367})e_2 \\
& + (e^{157} - e^{146} + e^{245} + e^{267})e_3 \\
& + (e^{127} + e^{136} - e^{235} - e^{567})e_4 \\
& + (e^{126} - e^{137} + e^{234} + e^{467})e_5 \\
& + (-e^{125} - e^{134} - e^{237} - e^{457})e_6 \\
& + (-e^{124} + e^{135} + e^{236} + e^{456})e_7, \\
\\
\psi = & (e^{23} + e^{45} + e^{67})e_1 \\
& + (e^{46} - e^{57} - e^{13})e_2 \\
& + (e^{12} - e^{47} - e^{56})e_3 \\
& + (e^{37} - e^{15} - e^{26})e_4 \\
& + (e^{14} + e^{27} + e^{36})e_5 \\
& + (e^{24} - e^{17} - e^{35})e_6 \\
& + (e^{16} - e^{25} - e^{34})e_7.
\end{aligned}$$

Next, we study the deformations of HL submanifolds in a G_2 manifold. Recall that McLean studied the deformations of compact special Lagrangian submanifolds in Calabi-Yau manifolds and proved the following theorem [10].

Theorem 2.9. *The moduli space of all deformations of a smooth, compact, orientable special Lagrangian submanifold L in a Calabi-Yau manifold N within the class of special Lagrangian submanifolds is a smooth manifold of dimension equal to $\dim(H^1(L))$.*

In this section, we provide the analogue of this theorem for HL manifolds (and also for RS manifolds in the next section), but in these cases, the space of deformations will be infinite-dimensional. Note that this is similar to the space of deformations of Lagrangian submanifolds of symplectic manifolds.

We first describe the normal bundle of an HL submanifold inside G_2 manifold M . Recall that an orthonormal 3-frame field $\Gamma = \langle u, v, w \rangle$ on (M, φ) is called a G_2 -frame field if $\varphi(u, v, w) = \langle u \times v, w \rangle = 0$, [1]. By results of Emery Thomas, [14], there exists a nonvanishing 2-frame field $\Lambda = \langle u, v \rangle$ on M .

Let $TM = \mathbb{E} \oplus \mathbb{V}$ be the corresponding splitting with $\mathbb{E} = \langle u, v, u \times v \rangle$. Let w be a unit section of the bundle $\mathbb{V} \rightarrow M$. Even though this section w may not exist on the entire manifold M , it exists on a tubular neighbourhood of the 3-skeleton $M^{(3)}$ of M (by obstruction theory) which is the complement of a 3 complex $Y \subset M$. Therefore, $\varphi(u, v, w) = \langle u \times v, w \rangle = 0$, and hence $\Gamma = \langle u, v, w \rangle$ is a G_2 (i.e. HL) frame field.

Consider the non-vanishing vector field

$$R = \chi(u, v, w) = -u \times (v \times w).$$

We recall the following lemma from [1].

Lemma 2.10. *The following properties hold:*

- (a) *If $\langle u, v, w \rangle$ is an HL 3-plane field, then $\mathbb{V} = \langle u, v, w, R \rangle$ is a coassociative 4-plane field.*
- (b) *$\mathbb{E} = \langle u \times v, v \times w, w \times u \rangle$ is an associative 3-plane field.*
- (c) *$\mathbb{E} \perp \mathbb{V}$.*
- (d) *$\{u, v, w, R, u \times v, v \times w, w \times u\}$ is an orthonormal frame field on M .*

In particular we can express φ as

$$\begin{aligned} \varphi = & u^\# \wedge v^\# \wedge (u^\# \times v^\#) + v^\# \wedge w^\# \wedge (v^\# \times w^\#) + w^\# \wedge u^\# \wedge (w^\# \times u^\#) \\ & + u^\# \wedge R^\# \wedge (v^\# \times w^\#) + v^\# \wedge R^\# \wedge (w^\# \times u^\#) + w^\# \wedge R^\# \wedge (u^\# \times v^\#) \\ & - (u^\# \times v^\#) \wedge (v^\# \times w^\#) \wedge (w^\# \times u^\#). \end{aligned}$$

Note that $\{u, v, w, R, u \times v, v \times w, w \times u\}$ and $\{u, v, w, R, w \times R, u \times R, v \times R\}$ are equivalent frames, therefore φ can be given by

$$\begin{aligned} \varphi = & u^\# \wedge v^\# \wedge (w^\# \times R^\#) + v^\# \wedge w^\# \wedge (u^\# \times R^\#) + w^\# \wedge u^\# \wedge (v^\# \times R^\#) \\ & + u^\# \wedge R^\# \wedge (u^\# \times R^\#) + v^\# \wedge R^\# \wedge (v^\# \times R^\#) + w^\# \wedge R^\# \wedge (w^\# \times R^\#) \\ & - (w^\# \times R^\#) \wedge (u^\# \times R^\#) \wedge (v^\# \times R^\#). \end{aligned}$$

The normal bundle of an HL submanifold can be decomposed as

$$N(HL) = \tilde{N}(HL) \oplus R,$$

where $\tilde{N}(HL)$ is generated by vector fields $u \times R$, $v \times R$, and $w \times R$. Since $\langle \psi(u, v), w \rangle = \varphi(u, v, w) = \langle u \times v, w \rangle = 0$ for an HL submanifold, $\tilde{N}(HL)$ is isomorphic to $T(HL)$. The cross product structure \times (also known as ψ) induces this isomorphism.

Theorem 2.11. *The space of infinitesimal deformations of a smooth, compact, orientable 3-dimensional Harvey Lawson submanifold HL in a G_2 manifold M within the class of HL submanifolds is infinite-dimensional. The deformation space can be identified with the direct sum of the spaces of smooth functions and closed 2-forms on HL.*

Proof. For a small vector field V the deformation map is a map, F , defined from the space of sections of the normal bundle, $\Gamma(N(HL))$, to the space of differential 3-forms, $\Lambda^3 T^*(HL)$ on HL , such that

$$\begin{aligned} F : \Gamma(N(HL)) &\rightarrow \Lambda^3 T^*(HL), \\ F(V) &= ((\exp_V)^*(\varphi|_{HL_V})). \end{aligned}$$

The deformation map F is the restriction of φ to HL_V and then pulled back to HL via $(\exp_V)^*$ where \exp_V is the normal exponential map which gives a diffeomorphism of HL onto its image HL_V in a neighborhood of 0.

There is a natural identification of normal vector fields to HL with differential 1-forms on HL . Furthermore, since HL is compact, these normal vector fields can

be identified with nearby submanifolds. Under these identifications, the kernel of F then corresponds to the HL deformations.

The linearization of F at (0) is given by

$$dF(0) : \Gamma(N(HL)) \rightarrow \Lambda^3 T^*(HL)$$

where

$$\begin{aligned} dF(0)(V) &= \frac{\partial}{\partial t} F(tV)|_{t=0} = \frac{\partial}{\partial t} [\exp_{tV}^*(\varphi)] \\ &= [\mathcal{L}_V(\varphi)|_{HL}]. \end{aligned}$$

Further, by Cartan's formula, we have

$$\begin{aligned} dF(0)(V) &= ((i_V d\varphi + d(i_V \varphi))|_{HL}) \\ &= d(i_V \varphi)|_{HL} = (d \star v), \end{aligned}$$

where i_V represents the interior derivative, v is the dual 1-form to the vector field V with respect to the induced metric, and $\star v$ is the Hodge dual of v on HL . Hence

$$dF(0)(V) = (d \star v) = (d^* v).$$

Therefore the space of nontrivial deformations of HL submanifolds can be identified with closed 2-forms on HL . The other component of the deformation space comes from the trivial deformations of HL submanifolds. These correspond to deforming a 3-dimensional HL manifold inside a coassociative manifold. By definition, any such 3-manifold will be HL , which implies that deformations of HL inside a coassociative submanifold in the direction of R can be identified with smooth functions on HL . □

3. DEFORMATIONS OF RS SUBMANIFOLDS

Let (M, φ) be a G_2 manifold. In this section, we define Lagrangian-type 4-dimensional submanifolds of M and describe their deformation space. Note that we can write the calibration 4-form $\star\varphi$, in local coordinates as

$$\star\varphi = e^{4567} + e^{2367} + e^{2345} + e^{1357} - e^{1346} - e^{1256} - e^{1247}.$$

Definition 3.1. An *RS manifold* is a 4-dimensional submanifold $RS \subset M$ of a G_2 manifold such that

$$\star\varphi|_{RS} = 0.$$

As with Harvey-Lawson manifolds, there is a tangent bundle-valued 4-form σ defining RS such that

$$\star\varphi|_{RS} = 0$$

if and only if

$$\langle \sigma|_{RS}, \sigma|_{RS} \rangle = 1.$$

This form, σ , is called the *coassociator*, and we can write it as

$$\sigma(u, v, w, z) = \langle v, w \times z \rangle u + \langle w, u \times z \rangle v + \langle u, v \times z \rangle w + \langle v, u \times w \rangle z.$$

In [9], Harvey and Lawson prove the coassociator equality,

$$*\varphi(u, v, w, z)^2 + \frac{1}{4} |\sigma(u, v, w, z)|^2 = |u \wedge v \wedge w \wedge z|^2.$$

Note that this identity provides justification for the above definition. Using the coordinates above for φ , we can write σ in coordinates as follows:

$$\begin{aligned}\sigma = & (-e^{1347} - e^{1356} - e^{1257} + e^{1246})e_1 \\ & + (-e^{2347} - e^{2356} - e^{1267} - e^{1245})e_2 \\ & + (-e^{2346} + e^{2357} - e^{1367} - e^{1345})e_3 \\ & + (e^{3456} + e^{2457} - e^{1467} - e^{1234})e_4 \\ & + (-e^{3457} + e^{2456} - e^{1567} - e^{1235})e_5 \\ & + (-e^{3467} - e^{2567} - e^{1456} - e^{1236})e_6 \\ & + (e^{3567} - e^{2467} - e^{1457} - e^{1237})e_7.\end{aligned}$$

Remark 3.2. Recall that the direct product $N^6 \times S^1$ is a G_2 manifold, where (N, ω, Ω) is a complex 3-dimensional Calabi-Yau manifold with Kähler form ω and nonvanishing holomorphic $(3, 0)$ -form Ω . In this case, $\varphi = \text{Re } \Omega + \omega \wedge dt$ and $*\varphi = -dt \wedge \text{Im } \Omega + \frac{1}{2}(\omega \wedge \omega)$. Therefore, the 4-manifolds given by $SL \times S^1$, where SL is a special Lagrangian submanifold of N with phase $\theta = 0$ will be RS submanifolds of $N^6 \times S^1$. Analogous results hold for the noncompact G_2 manifold $N^6 \times \mathbb{R}$.

Given a nonvanishing 4-frame field $\langle u, v, w, z \rangle$, note that we can define a nonvanishing vector field S given by

$$S = \sigma(u, v, w, z) = \langle v, w \times z \rangle u + \langle w, u \times z \rangle v + \langle u, v \times z \rangle w + \langle v, u \times w \rangle z.$$

Using

$$\langle \chi(u, v, w), z \rangle = *\varphi(u, v, w, z)$$

and

$$\langle \psi(u, v), w \rangle = \varphi(u, v, w) = \langle u \times v, w \rangle$$

we have the following lemma. Note that given a nonvanishing 2-frame field $\langle u, v \rangle$, we can extend it by some nonvanishing vector field w . Then $\tilde{S} := \sigma(u, v, u \times v, w)$ is a nonvanishing vector field on M .

Lemma 3.3. *Let $\tilde{S} = \sigma(u, v, u \times v, w)$ for nonvanishing vector fields $u, v, w \in TM$. Then the following properties hold:*

- (a) $\langle u, v, u \times v, w \rangle$ is an RS 4-plane field.
- (b) $\langle u, v, u \times v, \tilde{S} \rangle$ is an RS 4-plane field.
- (c) $\langle \tilde{S}, u \times \tilde{S}, v \times \tilde{S}, (u \times v) \times \tilde{S} \rangle$ is a coassociative 4-plane field.
- (d) $\langle u \times \tilde{S}, v \times \tilde{S}, (u \times v) \times \tilde{S} \rangle$ is an HL 3-plane field.
- (e) $\{u, v, u \times v, \tilde{S}, u \times \tilde{S}, v \times \tilde{S}, (u \times v) \times \tilde{S}\}$ is an orthonormal frame field.

The proof of this lemma is similar to that for HL manifolds, which can be found in [1]. Using these structures we can express the 3-form φ as

$$\begin{aligned}\varphi = & u^\# \wedge v^\# \wedge ((u \times v)^\# \times S^\#) + v^\# \wedge (u \times v)^\# \wedge (u \times S)^\# + (u \times v)^\# \wedge u^\# \wedge (v^\# \times S^\#) \\ & + u^\# \wedge S^\# \wedge (u^\# \times S^\#) + v^\# \wedge S^\# \wedge (v^\# \times S^\#) + (u \times v)^\# \wedge S^\# \wedge ((u \times v)^\# \times S^\#) \\ & - ((u \times v)^\# \times S^\#) \wedge (u^\# \times S^\#) \wedge (v^\# \times S^\#).\end{aligned}$$

The tangent bundle of an RS manifold decomposes as

$$T(RS) = \tilde{T}(RS) \oplus \tilde{S},$$

where $\tilde{T}(RS)$ is generated by vectors $\tilde{S} \times u$, $\tilde{S} \times v$, $\tilde{S} \times w$. Since

$$\langle \chi(u, v, w), z \rangle = \star \varphi(u, v, w, z) = 0$$

for an RS submanifold, its normal bundle $N(RS)$ is isomorphic to $\tilde{T}(RS)$. The bundle-valued 3-form χ induces this isomorphism.

Under these identifications, we can prove the following theorem.

Theorem 3.4. *The space of all infinitesimal deformations of a smooth, compact, orientable 4-dimensional RS submanifold in a G_2 manifold M within the class of RS submanifolds is infinite-dimensional and can be identified with closed differential 3-forms on RS .*

Proof. As in the proof of Theorem 2.11, for a small vector field V , the deformation map F is defined as

$$\begin{aligned} F : \Gamma(N(RS)) &\rightarrow \Lambda^4 T^*(RS) \\ F(V) &= (\exp_V)^*(\star \varphi) \end{aligned}$$

In other words, the deformation map F is the restriction of $\star \varphi$ to RS_V and then pulled back to RS via $(\exp_V)^*$ where \exp_V is the exponential map which gives a diffeomorphism of RS onto its image RS_V in a neighborhood of 0. For reasons analogous to those for HL manifolds, the kernel of F corresponds to deformations of RS manifolds.

Furthermore, we can write the linearization of F at (0) as

$$dF(0) : \Gamma(N(RS)) \rightarrow \Lambda^4 T^*(RS)$$

where

$$\begin{aligned} dF(0)(V) &= [-\mathcal{L}_V(\star \varphi)|_{RS}] \\ &= d(i_V(\star \varphi))|_{RS} = d \star v \end{aligned}$$

Here again, i_V represents the interior derivative and v is the dual 1-form to the vector field V with respect to the induced metric. Hence

$$dF(0)(V) = d \star v = \star(d^* v).$$

Therefore the space of infinitesimal deformations of RS manifolds can be identified with closed 3 forms on RS . □

Remark 3.5. Recall that for a Lagrangian submanifold L of (N^{2n}, ω) , there exists an almost complex structure J on N such that J maps vectors on L to vectors orthogonal to L . In other words,

$$\langle Ju, v \rangle = 0$$

for all $u, v \in TL$. As a final remark, we note that we have an analogous situation here for HL and RS manifolds. In the Harvey-Lawson case, the cross product \times acts as this complex structure. Further, in the case of RS manifolds, the triple cross product assumes the role of the complex structure, taking tangent vectors on the RS manifold to normal vectors.

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